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Local quasi-interpolation by cubic C^1 splines on type-6 tetrahedral partitions

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We describe an approximating scheme based on cubic C^1 splines on type-6 tetrahedral partitions using data on volumetric grids. The quasi-interpolating piecewise polynomials are directly determined by setting their Bernstein–Bézier coefficients to appropriate combinations of the data values. Hence, each polynomial piece of the approximating spline is immediately available from local portions of the data, without using prescribed derivatives at any point of the domain. The locality of the method and the uniform boundedness of the associated operator provide an error bound, which shows that the approach can be used to approximate and reconstruct trivariate functions. Simultaneously, we show that the derivatives of the quasi-interpolating splines yield nearly optimal approximation order. Numerical tests with up to 17×10^6 data sites show that the method can be used for efficient approximation.

Keywords: trivariate splines; quasi-interpolation; Bernstein–Bézier form; type-6 tetrahedral partitions; approximation order.

1. Introduction

We investigate the problem of constructing appropriate non-discrete models from given discrete data on volumetric grids. The development of such trivariate models approximating given data is important because it is the theoretical basis for many applications, such as scientific visualization, medical imaging and numerical simulation. A standard example is trilinear interpolation, i.e. roughly speaking, interpolation at the volumetric grid points based on the straightforward tensor-product extension of univariate linear interpolating splines. The simplicity of this local spline model and the fact that it approximates sufficiently smooth functions with order two are the main theoretical reasons for its frequent use in the above mentioned applications (cf. Bajaj, 1999; Marschner & Lobb, 1994; Meissner *et al.*, 2000; Parker *et al.*, 1998; Theußl *et al.*, 2002). Although based on piecewise cubic polynomials, this continuous model is not smooth, while it is known that the existence and certain approximation properties of the derivatives would be advantageous for practical purposes. On the other hand, tri-quadratic and tricubic tensor-product splines can be used to construct smooth models of the data. However, these are piecewise polynomials of higher degrees—six and nine, respectively, and both schemes usually require (approximate) derivative data at certain prescribed points. This raises the natural problem of constructing an alternative, local and smooth spline model based on piecewise cubic polynomials, which uses only

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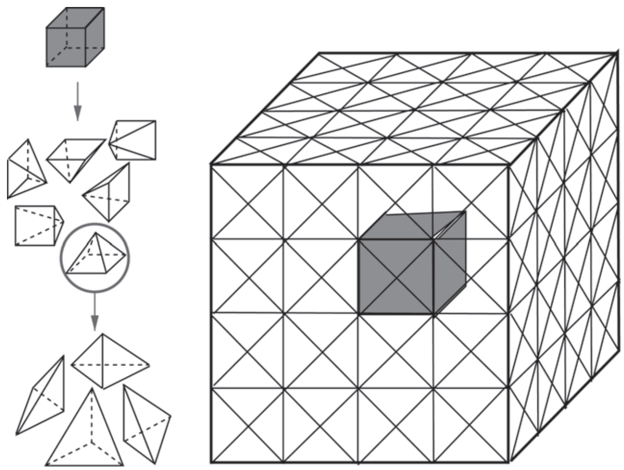

FIG. 1. Type-6 tetrahedral partitions Δ are obtained from subdividing each box into 24 congruent tetrahedra (left). The restrictions to certain planes parallel to the three coordinate planes are four-directional meshes (right).

data values on the volumetric grid (no prescribed derivatives), and simultaneously approximates smooth functions as well as their derivatives. The aim of this paper is to develop a quasi-interpolation method, which is the first solution with these properties.

Our approach is based on the piecewise Bernstein–Bézier form (BB-form) of cubic C^1 splines on type-6 tetrahedral partitions. These are uniform tetrahedral partitions of the 3D domain obtained by subdividing boxes into 24 tetrahedra using six planes (see Fig. 1). Smooth splines on these types of partitions have recently gained some interest in multivariate spline theory. A structural analysis for any degree was given in Hangelbroek *et al.* (2004), while macro-element constructions based on degree five and six super-splines were developed in Lai & Le Méhauté (2004) and Schumaker & Sorokina (2005). One of the reasons for the interest is that the type-6 tetrahedral partition is a natural 3D analogue of the well-known bivariate four-directional mesh (Chui, 1988; Davydov & Zeilfelder, 2004; Lai & Schumaker, 2006; Nürnberger & Zeilfelder, 2000; Sablonnière, 2003; Sorokina & Zeilfelder, 2005). Another reason results from a comparison with smooth trivariate splines on more general partitions (Alfeld, 1984; Alfeld & Schumaker, 2005a,b,c; Alfeld *et al.*, 1992, 1993; Lai & Schumaker, 2006; Nürnberger *et al.*, 2005b; Sorokina, 2004; Sorokina & Worsey, submitted; Worsey & Farin, 1987; Worsey & Piper, 1988). These spaces are usually extremely complex; however, the structure of the spaces over uniform type partitions (cf. Hangelbroek *et al.*, 2004; Hecklin *et al.*, 2006; Schumaker & Sorokina, 2004, 2005) is somewhat simplified, sometimes providing the possibility of using lower-degree smooth splines. This is important from a practical point of view for the above mentioned applications. However, satisfying all smoothness conditions in the various spatial directions still remains complicated.

We take advantage of the uniform structure of type-6 tetrahedral partitions, which implies that certain Bernstein–Bézier coefficients (BB-coefficients) of the C^1 splines have to be simple averages of certain other BB-coefficients. Roughly speaking, the basic idea of the new approach is as follows. Given data on a volumetric grid, we consider appropriate local portions of the data and set each BB-coefficient of the approximating cubic spline directly by applying some natural and simple averaging rules. The setting of the BB-coefficients is done in such a way that the C^1 smoothness of the resulting spline is guaranteed, while certain approximation properties hold. This procedure allows an efficient computation of the splines, and since we only need data values to perform the algorithm, (approximate) derivatives are not required at any point in the 3D domain. This stands in contrast to certain finite-element approaches, and some more recent macro-element methods as, for instance, the quintic C^1 Hermite interpolant in Alfeld (1984). Another difference from the approaches in Alfeld (1984), Alfeld & Schumaker (2005a,b,c), Nürnberger *et al.* (2005b), Worsey & Farin (1987) and Worsey & Piper (1988) is that in our method, we do not split any tetrahedra of the underlying partition. Moreover, by the nature of our algorithm, the splines are directly available from the data. Hence, neither the construction of (minimal) determining sets (cf. Lai & Schumaker, 2006; Nürnberger *et al.*, 2005b; Schumaker & Sorokina, 2004, 2005) nor an intermediate step making use of certain locally supported splines is needed in our approach. It falls into the class of quasi-interpolation methods for trivariate splines. More precisely, we show that the scheme is associated with a quasi-interpolation operator, which is uniformly bounded, local and satisfies certain reproduction properties. Using this, we can guarantee the stability of the approach and we derive error bounds, which show that the splines and their derivatives simultaneously approximate data coming from smooth functions. As a non-standard phenomenon in approximation theory, we observe that the approximation order of the splines and their derivatives is the same, which in the latter case is nearly optimal. Although we use the BB-form mainly as a theoretical tool here, it should be pointed out that the Bernstein–Bézier techniques (BB-techniques) also play an important role in a practical context since they allow efficient representation, computation, evaluation and visualization of the splines (cf. de Boor, 1987; Farin, 1986; Nürnberger *et al.*, 2005a; Rössl *et al.*, 2004; Schlosser *et al.*, 2005). In fact, we use these techniques in the implementations of our method.

The paper is organized as follows: In Section 2, we give some preliminaries on the BB-form of cubic C^1-splines on type-6 tetrahedral partitions and describe the smoothness conditions of the spaces in a convenient form. Section 3 is devoted to our quasi-interpolating scheme. In Section 4, we show that the scheme leads to cubic splines which are globally smooth. Some useful properties of the corresponding quasi-interpolation operator are discussed in Section 5. These results are used in Section 6, where we provide error bounds for the quasi-interpolating splines and their derivatives. In Section 7, we provide numerical tests involving up to 17×10^6 samples, and conclude the paper with several remarks.

2. BB-form of cubic C^1 splines on type-6 partitions

In this preliminary section, we define type-6 tetrahedral partitions Δ, recall some facts on the piecewise BB-form of trivariate cubic splines on Δ and describe the C^1 smoothness conditions for these spaces.

For a given grid size $h > 0$ and integers n, m, r, let

$$V := \{v_{ijk} = (ih, jh, kh), i = 0, \dots, n+1, j = 0, \dots, m+1, k = 0, \dots, r+1\}$$

be the set of $(n+2) \times (m+2) \times (r+2)$ grid points. Each interior grid point v_{ijk}, for $i \notin \{0, n+1\}$, $j \notin \{0, m+1\}$ and $k \notin \{0, r+1\}$, lies in the centre of the box

$$Q_{ijk} := [(2i-1)h/2, (2i+1)h/2] \times [(2j-1)h/2, (2j+1)h/2] \times [(2k-1)h/2, (2k+1)h/2],$$

and the collection of these boxes forms a partition

$$\diamondsuit := \{Q_{ijk}: i = 1, \dots, n, j = 1, \dots, m, k = 1, \dots, r\}$$

of the rectangular, volumetric domain

$$\Omega = [h/2, (2n+1)h/2] \times [h/2, (2m+1)h/2] \times [h/2, (2r+1)h/2]$$
$$\subseteq [0, (n+1)h] \times [0, (m+1)h] \times [0, (r+1)h] =: \tilde{\Omega}.$$

In the following, we call the boxes Q_{ijk} with $i \in \{1, n\}$ or $j \in \{1, m\}$ or $k \in \{1, r\}$ boundary boxes, and the square faces of these boxes which lie on the boundary of Ω boundary square faces. Otherwise, we call them interior boxes and interior square faces, respectively.

For building a suitable tetrahedral partition from $\diamondsuit$, we subdivide the boxes as follows. In each box Q_{ijk}, we first draw in the two diagonals of each of the six faces of Q_{ijk}, and then connect the centre v_{ijk} of Q_{ijk} with the eight vertices of Q_{ijk} as well as with the centres of the six faces of Q_{ijk}. Now each box can be considered as being subdivided into six pyramids, where each pyramid is further decomposed in four tetrahedra of the same form, see Fig. 1. Hence, this procedure splits each box Q_{ijk} into 24 congruent tetrahedra yielding a tetrahedral partition $\triangle$ of Ω, which consists of $24 \times n \times m \times r$ tetrahedra. This partition $\triangle$ is called a type-6 tetrahedral partition because it is alternatively described as the result of slicing each box $Q_{ijk} \in \diamondsuit$ with six different planes. The restriction of $\triangle$ to any plane E containing a face of some box in $\diamondsuit$ is a (bivariate) four-directional mesh of the intersecting square domain $E \cap \Omega$ (see Fig. 1, right). Therefore, Δ can be considered as a trivariate generalization of the four-directional mesh well-known in the bivariate spline theory.

In the following, we consider the space of cubic C^1 splines on $\triangle$ defined as

$$\mathcal{S} = \{s \in C^1(\Omega)\colon s|_T \in \mathcal{P}_3, \text{ for all } T \in \triangle\}, \tag{2.1}$$

where $C^1(\Omega)$ is the set of continuously differentiable functions on Ω, and $\mathcal{P}_3$ denotes the space of trivariate polynomials of total degree three. Note that it is obvious that $\mathcal{S}$ is a subspace of a simpler space of cubic continuous splines on $\triangle$ defined by $\mathcal{S}_0 = \{s \in C(\Omega)\colon s|_T \in \mathcal{P}_3, \text{ for all } T \in \triangle\}$, where $C(\Omega)$ denotes the set of continuous functions on Ω.

Throughout the paper, we use the piecewise BB-form of the splines on Δ (cf. de Boor, 1987; Chui, 1988; Farin, 1986; Lai & Schumaker, 2006; Nürnberger & Zeilfelder, 2000). Given a spline $s \in \mathcal{S}$, each of its polynomial pieces $p = s|_T \in \mathcal{P}_3$ on a tetrahedron $T = \langle v_0, v_1, v_2, v_3\rangle \in \triangle$ with vertices v_0, v_1, v_2 and v_3 is determined by

$$p = \sum_{i+j+k+\ell=3} c_{ijk\ell} B^T_{ijk\ell}, \tag{2.2}$$

where

$$B^T_{ijk\ell} = \frac{6}{i!\,j!\,k!\,\ell!} b_0^i b_1^j b_2^k b_3^\ell$$

are the cubic Bernstein polynomials on T for $i+j+k+\ell = 3$. Here, b_0, b_1, b_2 and b_3 denote the barycentric coordinates with respect to T, that are linear trivariate polynomials determined by

$$b_i(v_j) = \delta_{i,j}, \quad j = 0, \ldots, 3,$$

where $\delta_{i,j}$ is the Kronecker's symbol. It is easy to see that the Bernstein polynomials form the partition of unity

$$\sum_{i+j+k+\ell=3} B^T_{ijk\ell} = 1. \tag{2.3}$$

As usual, we associate the BB-coefficient $c_{ijk\ell}$ of p in the form (2.2) with the domain point

$$\xi_{ijk}\ell := (iv_0 + jv_1 + kv_2 + \ell v_3)/3, \quad i + j + k + \ell = 3,$$

in T, and we let $\mathcal{D}_\triangle$ be the union of the sets of domain points associated with the tetrahedra of $\triangle$. For later use, we also mention that if we denote by $p_{ijk\ell}$ the value of $p \in \mathcal{P}_3$ at the domain point $\xi_{ijk\ell}$, $i + j + k + \ell = 3$, in T, then the unique Lagrange polynomial interpolation at these 20 points gives

$$\begin{aligned} c_{3000} &= p_{3000}, \quad c_{2100} = \frac{1}{3}p_{0300} - \frac{5}{6}p_{3000} + 3p_{2100} - \frac{3}{2}p_{1200}, \\ c_{1110} &= \frac{1}{3}(p_{3000} + p_{0300} + p_{0030}) + \frac{9}{2}p_{1110} \\ &\quad - \frac{3}{4}(p_{2100} + p_{1200} + p_{2010} + p_{1020} + p_{0210} + p_{0120}), \end{aligned} \tag{2.4}$$

with similar formulae for the remaining BB-coefficients of p in its representation (2.2) with respect to T.

A well-known advantage of the BB-form is that it conveniently allows to describe smoothness conditions (cf. de Boor, 1987; Chui, 1988; Farin, 1986) between the polynomial pieces of the splines on neighbouring tetrahedra. Given any two (non-degenerate) tetrahedra $T = \langle v_0, v_1, v_2, v_3\rangle$, $\tilde{T} = \langle v_0, v_1, v_2, \tilde{v}_3\rangle$ sharing a common triangular face $T \cap \tilde{T} = \langle v_0, v_1, v_2\rangle$, let s be a cubic continuous spline on $T \cup \tilde{T}$ in its piecewise BB-form (2.2): $s|_T = p$ and $s|_{\tilde{T}} = \tilde{p}$ with the corresponding BB-coefficients $c_{ijk\ell}$ and $\tilde{c}_{ijk\ell}$, respectively. Then s is C^1 smooth across $T \cap \tilde{T}$ if and only if

$$c_{ijk1} = \tilde{c}_{i+1jk0}b_0(\tilde{v}_3) + \tilde{c}_{ij+1k0}b_1(\tilde{v}_3) + \tilde{c}_{ijk+1\,0}b_2(\tilde{v}_3) + \tilde{c}_{ijk1}b_3(\tilde{v}_3), \tag{2.5}$$

where $i + j + k = 2$. In general, there are five coefficients involved in each of these six conditions. In Fig. 2 (right), the common triangular face $T \cap \tilde{T}$ is shaded grey, the domain points associated with the BB-coefficients involved in the smoothness conditions are shown as grey dots and the conditions are illustrated by using thick lines and small tetrahedra with thick boundary lines. If one or two of the barycentric coordinates of the point $\tilde{v}_3$ vanish, then the number of the involved coefficients is four and three, respectively. In these cases, the smoothness conditions degenerate to lower-dimensional conditions which are similar to those in the bivariate and univariate setting. Figure 2 (left) shows such an example, where two barycentric coordinates of $\tilde{v}_3$ are zeros, and hence, the smoothness conditions are similar to the univariate ones.

Throughout the paper, we consider type-6 tetrahedral partitions. The C^1 smoothness conditions for the splines from $\mathcal{S}$ can be described in a very simple way. We observe that for these tetrahedral partitions, only two types of smoothness conditions illustrated in Fig. 2 do appear, and that for each of these conditions, the weights in (2.5) —the barycentric coordinates of the point $\tilde{v}_3$—are always the same rational numbers. Therefore, we may consider these smoothness conditions (2.8)–(2.9) as simple averaging rules to be satisfied by the BB-coefficients of the cubic C^1 splines from $\mathcal{S}$. More precisely, according to the uniform structure of $\triangle$, every tetrahedron $T = \langle v_0, v_1, v_2, v_3\rangle \in \triangle$ with vertices v_0, v_1, v_2 and v_3 has one vertex, say v_0, at the centre of a box Q_{ijk}, another vertex, say v_1, at the midpoint of one of the faces of Q_{ijk} and the two remaining vertices, v_2 and v_3, are vertices of Q_{ijk}. It suffices to describe the C^1 smoothness conditions across the four (interior) triangular faces $\langle v_0, v_1, v_2\rangle$, $\langle v_0, v_1, v_3\rangle$, $\langle v_0, v_2, v_3\rangle$ and $\langle v_1, v_2, v_3\rangle$ of T. By using (2.5) and some elementary computations, we obtain that $s \in \mathcal{S}$ if and

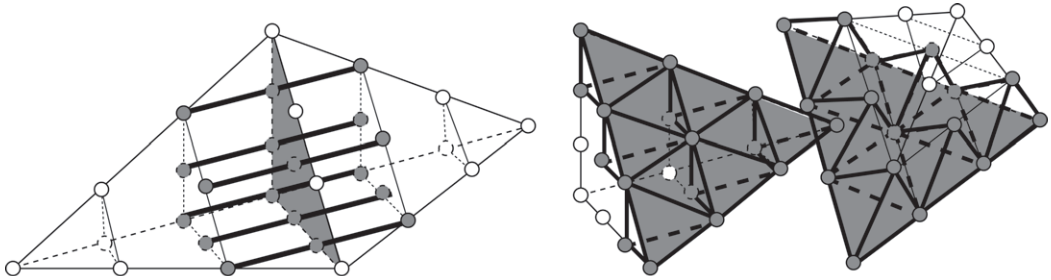

FIG. 2. Illustration of the six smoothness conditions determined by (2.6)–(2.7) (left) and (2.9) (right). Smoothness conditions across the common triangular face of two neighbouring tetrahedra which degenerate to univariate smoothness conditions (with three coefficients involved in each condition) are shown on the left, while the non-degenerate case (with five coefficients involved in each condition) is shown on the right. In both cases, the BB-coefficients associated with domain points shown as white dots are not involved in any smoothness conditions across the shaded triangular face, while the remaining BB-coefficients (shown as grey dots) are involved in such conditions.

only if the following conditions for its BB-coefficients in the representation (2.2) are satisfied:

- Smoothness across $F = \langle v_0, v_1, v_2\rangle$.
  Let $\tilde{T} = \langle v_0, v_1, v_2, \tilde{v}_3\rangle$ be the tetrahedron sharing the face F with T. Then,

$$\begin{aligned} c_{0120} &= (c_{0021} + \tilde{c}_{0021})/2, & c_{0210} &= (c_{0111} + \tilde{c}_{0111})/2, \\ c_{0300} &= (c_{0201} + \tilde{c}_{0201})/2, & c_{1110} &= (c_{1011} + \tilde{c}_{1011})/2, \\ c_{1200} &= (c_{1101} + \tilde{c}_{1101})/2, & c_{2100} &= (c_{2001} + \tilde{c}_{2001})/2. \end{aligned} \tag{2.6}$$

- Smoothness across $F = \langle v_0, v_1, v_3\rangle$.
  Let $\tilde{T} = \langle v_0, v_1, \tilde{v}_2, v_3\rangle$ be the tetrahedron sharing the face F with T. Then,

$$\begin{aligned} c_{0102} &= (c_{0012} + \tilde{c}_{0012})/2, & c_{0201} &= (c_{0111} + \tilde{c}_{0111})/2, \\ c_{0300} &= (c_{0210} + \tilde{c}_{0210})/2, & c_{1101} &= (c_{1011} + \tilde{c}_{1011})/2, \\ c_{1200} &= (c_{1110} + \tilde{c}_{1110})/2, & c_{2100} &= (c_{2010} + \tilde{c}_{2010})/2. \end{aligned} \tag{2.7}$$

- Smoothness across $F = \langle v_0, v_2, v_3\rangle$.
  Let $\tilde{T} = \langle v_0, \tilde{v}_1, v_3, v_4\rangle$ be the tetrahedron sharing the face F with T. Then,

$$\begin{aligned} c_{3000} &= c_{2100} + \tilde{c}_{2100} - (c_{2010} + c_{2001})/2, \\ c_{2010} &= c_{1110} + \tilde{c}_{1110} - (c_{1020} + c_{1011})/2, \\ c_{2001} &= c_{1101} + \tilde{c}_{1101} - (c_{1002} + c_{1011})/2, \\ c_{1020} &= c_{0120} + \tilde{c}_{0120} - (c_{0030} + c_{0021})/2, \\ c_{1002} &= c_{0102} + \tilde{c}_{0102} - (c_{0003} + c_{0012})/2, \\ c_{1011} &= c_{0111} + \tilde{c}_{0111} - (c_{0012} + c_{0021})/2. \end{aligned} \tag{2.8}$$

- Smoothness across $F = \langle v_1, v_2, v_3\rangle$.

Let $\tilde{T} = \langle \tilde{v}_0, v_1, v_2, v_3 \rangle$ be the tetrahedron sharing the face F with T. Then,

$$\begin{aligned} c_{0120} &= (c_{1020} + \tilde{c}_{1020})/2, \quad & c_{0102} &= (c_{1002} + \tilde{c}_{1002})/2, \\ c_{0111} &= (c_{1011} + \tilde{c}_{1011})/2, \quad & c_{0210} &= (c_{1110} + \tilde{c}_{1110})/2, \\ c_{0201} &= (c_{1101} + \tilde{c}_{1101})/2, \quad & c_{0300} &= (c_{1200} + \tilde{c}_{1200})/2. \end{aligned} \tag{2.9}$$

Note that (2.6)–(2.8) characterize the smoothness conditions across the triangular faces completely contained in a box, while (2.9) describes the smoothness conditions across the triangular faces located between the neighbouring boxes.

Equations (2.6)–(2.9) show that for the splines from $\mathcal{S}$, smoothness conditions across common triangular faces of neighbouring tetrahedra in $\triangle$ are basically described by the means of two very simple formulae. However, if we consider a complete type-6 tetrahedral partition Δ, then satisfying all the conditions to obtain a globally C^1 smooth spline is often a complex task, because a huge number of conditions have to be simultaneously satisfied, and they cannot be considered independently. An analysis of these complex relations and different formulae for the number of degrees of freedom, i.e. the dimension of the space of C^1 splines of any degree on $\triangle$, are given in Hangelbroek *et al.* (2004). For the reader's convenience, we recall the following result for the particular case of cubic C^1 splines on $\triangle$.

THEOREM 2.1 The dimension of the spline space $\mathcal{S}$ is equal to

$$6nmr + 8(nm + nr + mr) + 6(n + m + r) + 4. \tag{2.10}$$

Note that in Hangelbroek *et al.* (2004), non-local arguments were needed to prove the general statement. However, the result of Theorem 2.1 already indicates that there is some hope that efficient local approximation operators based on $\mathcal{S}$ can be constructed. This stands in contrast to cubic C^1 splines on the so-called Freudenthal partitions, which are uniform type tetrahedral partitions also obtained from $\diamondsuit$, but where each box is subdivided into six subtetrahedra (see Hecklin *et al.*, 2006).

3. Quasi-interpolation scheme

In this section, we describe our quasi-interpolation scheme. Given $(m + 2) \times (n + 2) \times (r + 2)$ data values

$$f(v_{ijk}), \quad i = 0, \ldots, n+1, \quad j = 0, \ldots, m+1, \; k = 0, \ldots r+1, \tag{3.1}$$

of a continuous function f at the points v_{ijk} in V, our method is to set directly each BB-coefficient $c_\xi = c_\xi(s_f)$, $\xi \in \mathcal{D}_\triangle$, in the piecewise representation (2.2) of a cubic spline s_f on $\triangle$. For each domain point ξ in a tetrahedron T contained in $Q = Q_{ijk}$, we use the 27 values of f at the points $v_{i+i_0 j+j_0 k+k_0}$, where $i_0, j_0, k_0 \in \{-1, 0, 1\}$, and uniquely determine the BB-coefficient c_ξ of s_f by building appropriate averages of this local portion of the data. More precisely, we set

$$c_\xi := \sum_{i_0, j_0, k_0 \in \{-1,0,1\}} \omega_{i_0 j_0 k_0}(\xi) \, f\left(v_{i+i_0 j+j_0 k+k_0}\right), \tag{3.2}$$

where the non-negative weights $\omega_{i_0 j_0 k_0}(\xi)$ are independent of Q and T. In this way, we uniquely determine $s_f|_T$ for each T in Δ and, hence, the approach is completely symmetric. In the remaining part of this section, we describe the specific choice of the weights in (3.2) defining our scheme.

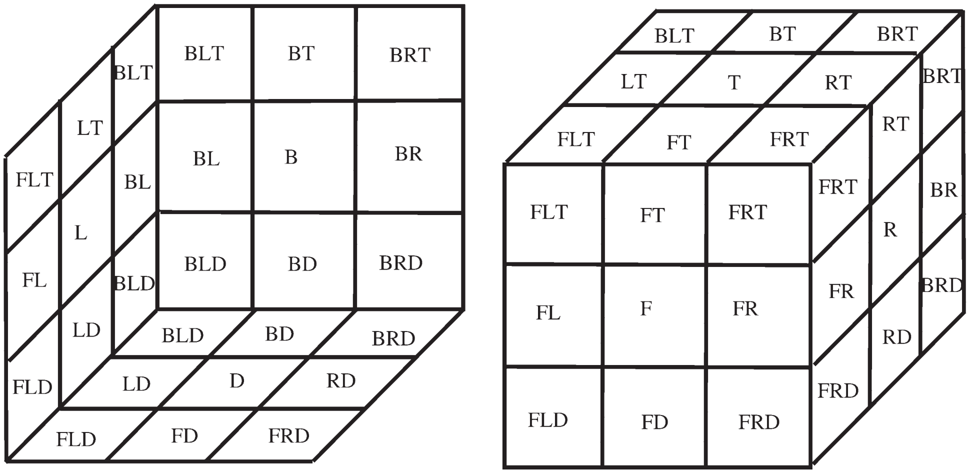

FIG. 3. The surface view of 26 boxes intersecting with an interior box $Q = Q_{ijk}$. I = box Q Itself, F = Front, B = Back, R = Right, L = Left, T = Top, and D = Down.

In order to keep these formulae and corresponding proofs as short as possible, and for a better geometric understanding, we introduce the following notation illustrated in Fig. 3. The value $f(v_{ijk})$ at the centre v_{ijk} of the box $Q := Q_{ijk}$. Itself is abbreviated by

$$I := f(v_{ijk}),$$

while the given values in its Front neighbouring box Q_{i-1jk}, Left neighbouring box Q_{ij-1k} and Down neighbouring box Q_{ijk-1}, respectively, are abbreviated by

$$F := f(v_{i-1jk}), \quad L := f(v_{ij-1k}) \quad \text{and} \quad D := f(v_{ijk-1}).$$

Similarly, we set

$$B := f(v_{i+1jk}), \quad R := f(v_{ij+1k}) \quad \text{and} \quad T := f(v_{ijk+1})$$

for the values in the Back, Right and Top neighbouring boxes of Q, respectively. For the given values in the Front-Left box $Q_{i-1j-1k}$, Front-Right box $Q_{i-1j+1k}$, Front-Down box $Q_{i-1jk-1}$ and Front-Top box $Q_{i-1jk+1}$ of Q, respectively, we use the abbreviations

$$\mathrm{FL} := f(v_{i-1j-1k}), \quad \mathrm{FR} := f(v_{i-1j+1k}),$$
$$\mathrm{FD} := f(v_{i-1jk-1}), \quad \mathrm{FT} := f(v_{i-1jk+1}),$$

and, similarly, we set

$$\mathrm{BL} := f(v_{i+1j-1k}), \quad \mathrm{BR} := f(v_{i+1j+1k}),$$
$$\mathrm{BD} := f(v_{i+1jk-1}), \quad \mathrm{BT} := f(v_{i+1jk+1}),$$
$$\mathrm{LD} := f(v_{ij-1k-1}), \quad \mathrm{LT} := f(v_{ij-1k+1}),$$
$$\mathrm{RD} := f(v_{ij+1k-1}), \quad \mathrm{RT} := f(v_{ij+1k+1})$$

for the values in its Back-Left, Back-Right, Back-Down, Back-Top, Left-Down, Left-Top, Right-Down, Right-Top boxes, respectively. Finally, the values in the Front-Left-Down box $Q_{i-1j-1k-1}$ and Front-Left-Top box $Q_{i-1j-1k+1}$, respectively, are denoted by

$$\mathrm{FLD} := f(v_{i-1j-1k-1}), \quad \mathrm{FLT} := f(v_{i-1j-1k+1}),$$

while we set

$$\begin{aligned} \mathrm{FRD} &:= f(v_{i-1j+1k-1}), \quad \mathrm{FRT} := f(v_{i-1j+1k+1}), \\ \mathrm{BLD} &:= f(v_{i+1j-1k-1}), \quad \mathrm{BLT} := f(v_{i+1j-1k+1}), \\ \mathrm{BRD} &:= f(v_{i+1j+1k-1}), \quad \mathrm{BRT} := f(v_{i+1j+1k+1}) \end{aligned}$$

for the values in its Front-Right-Down, Front-Right-Top, Back-Left-Down, Back-Left-Top, Back-Right-Down and Back-Right-Top boxes, respectively.

The type-6 tetrahedral partition $\triangle$ is symmetric in the sense that each tetrahedron in $\triangle$ has one vertex at the centre v_{ijk} of a box $Q = Q_{ijk}$ from $\diamondsuit$, another vertex at the centre of one of the faces of Q and two other vertices coincide with vertices of that face. Since our scheme is also completely symmetric, it suffices to consider the tetrahedron $T = \langle v_0, v_1, v_2, v_3 \rangle$, where $v_0 = v_{ijk}$ is the centre of Q, $v_1 = (v_{ijk} + v_{i-1jk})/2$ is the centre of the front face of Q and $v_2 = (v_{ijk} + v_{i-1j-1k+1})/2$, $v_3 = (v_{ijk} + v_{i-1j+1k+1})/2$ are the vertices of the upper edge of the front face (see Figs 4–9). Next we show how to set the BB-coefficients $c_{ijk\ell}$, $i + j + k + \ell = 3$, of $s_f|_T$ for that particular tetrahedron T. We consider four different layers

$$\mathcal{L}_i = \{\xi \in T : \xi = \xi_{ijk\ell},\ j + k + \ell = 3 - i\}, \quad i = 0, \ldots, 3,$$

of domain points in T. The formulae for the BB-coefficients of s_f associated with domain points in the remaining tetrahedra in Q (and, therefore, for all tetrahedra in $\triangle$) then immediately follow from symmetry.

Before providing explicit formulae, we exemplary describe the setting of two coefficients in simple terms for better geometric visualization of the masks and justification of our specific notation for data values. For instance, to obtain the BB-coefficient associated with the domain point located at the corner of a box, we simply average the data at the centres of eight boxes sharing this corner (cf. the first two formulae in (3.3), below). To obtain the BB-coefficient associated with the domain point located at the centre of the box Q_{ijk}, we take the following average: the datum at this point with weight 36, the data at the centres of the six boxes sharing faces with Q_{ijk} with weight 8 each and the data at the centres of the 12 remaining boxes sharing edges with Q_{ijk} (cf. formula (3.9), below).

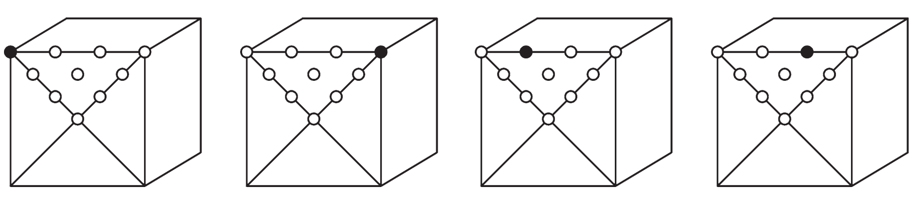

FIG. 4. Location of the domain points ξ_{0030}, ξ_{0003}, ξ_{0021}, ξ_{0012} in layer $\mathcal{L}_0$.

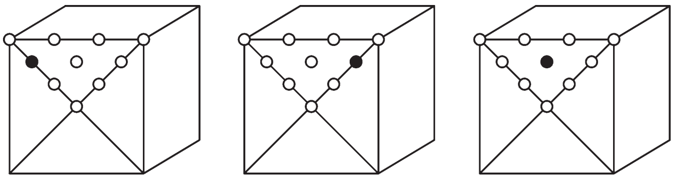

FIG. 5. Location of the domain points ξ_{0120}, ξ_{0102}, ξ_{0111} in layer $\mathcal{L}_0$.

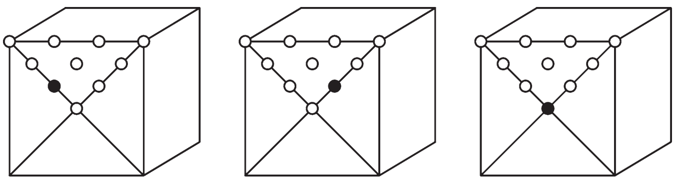

FIG. 6. Location of the domain points ξ_{0210}, ξ_{0201}, ξ_{0300} in layer $\mathcal{L}_0$.

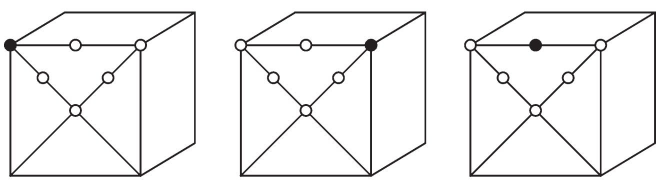

FIG. 7. Location of the domain points ξ_{1020}, ξ_{1002}, ξ_{1011} in layer $\mathcal{L}_1$.

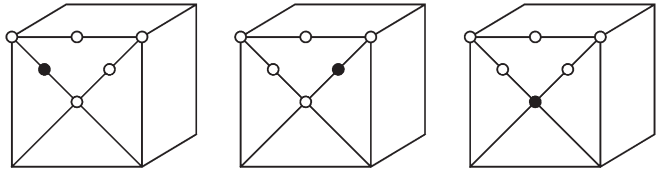

FIG. 8. Location of the domain points ξ_{1110}, ξ_{1101}, ξ_{1200} in layer $\mathcal{L}_1$.

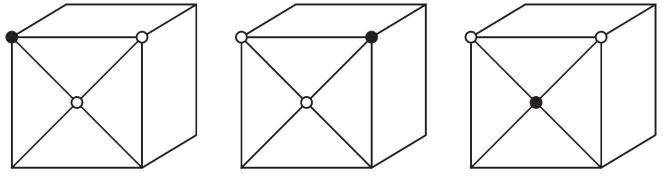

FIG. 9. Location of the domain points ξ_{2010}, ξ_{2001}, ξ_{2100} in layer $\mathcal{L}_2$.

Now we proceed with the explicit description of our quasi-interpolation scheme. For the BB-coefficients of $s_f|_T$ associated with points in $\mathcal{L}_0$ (see Figs 4–6), we set

$$
\begin{aligned}
c_{0030} &:= \frac{1}{8}(I + F + L + T + \mathrm{LT} + \mathrm{FL} + \mathrm{FT} + \mathrm{FLT}),\\
c_{0003} &:= \frac{1}{8}(I + F + R + T + \mathrm{RT} + \mathrm{FR} + \mathrm{FT} + \mathrm{FRT}),\\
c_{0021} &:= \frac{5}{24}(I + F + T + \mathrm{FT}) + \frac{1}{24}(L + \mathrm{FL} + \mathrm{LT} + \mathrm{FLT}),\\
c_{0012} &:= \frac{5}{24}(I + F + T + \mathrm{FT}) + \frac{1}{24}(R + \mathrm{FR} + \mathrm{RT} + \mathrm{FRT}),
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
c_{0120} &:= \frac{5}{24}(I + F) + \frac{1}{8}(L + T + \mathrm{FL} + \mathrm{FT}) + \frac{1}{24}(\mathrm{LT} + \mathrm{FLT}),\\
c_{0102} &:= \frac{5}{24}(I + F) + \frac{1}{8}(R + T + \mathrm{FR} + \mathrm{FT}) + \frac{1}{24}(\mathrm{RT} + \mathrm{FRT}),\\
c_{0111} &:= \frac{13}{48}(I + F) + \frac{7}{48}(T + \mathrm{FT}) + \frac{1}{32}(L + R + \mathrm{FL} + \mathrm{FR})\\
&\quad + \frac{1}{96}(\mathrm{LT} + \mathrm{RT} + \mathrm{FLT} + \mathrm{FRT}),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
c_{0210} &:= \frac{13}{48}(I + F) + \frac{17}{192}(L + T + \mathrm{FL} + \mathrm{FT})\\
&\quad + \frac{1}{96}(\mathrm{LT} + \mathrm{FLT}) + \frac{1}{64}(R + D + \mathrm{FR} + \mathrm{FD})\\
&\quad + \frac{1}{192}(\mathrm{RT} + \mathrm{LD} + \mathrm{FRT} + \mathrm{FLD}),\\
c_{0201} &:= \frac{13}{48}(I + F) + \frac{17}{192}(R + T + \mathrm{FR} + \mathrm{FT})\\
&\quad + \frac{1}{96}(\mathrm{RT} + \mathrm{FRT}) + \frac{1}{64}(L + D + \mathrm{FL} + \mathrm{FD})\\
&\quad \frac{1}{192}(\mathrm{RD} + \mathrm{LT} + \mathrm{FLT} + \mathrm{FRD}),\\
c_{0300} &:= \frac{13}{48}(I + F) + \frac{5}{96}(L + R + T + D + \mathrm{FL} + \mathrm{FR} + \mathrm{FT} + \mathrm{FD})\\
&\quad + \frac{1}{192}(\mathrm{RT} + \mathrm{RD} + \mathrm{LT} + \mathrm{LD} + \mathrm{FRT} + \mathrm{FRD} + \mathrm{FLT} + \mathrm{FLD}).
\end{aligned}
\tag{3.5}
$$

For the BB-coefficients of $s_f|_T$ associated with points in $\mathcal{L}_1$ (see Figs 7 and 8), we set

$$\begin{aligned} c_{1020} &:= \frac{1}{4}I + \frac{1}{6}(F + L + T) + \frac{1}{12}(\mathrm{LT} + \mathrm{FL} + \mathrm{FT}), \\ c_{1002} &:= \frac{1}{4}I + \frac{1}{6}(F + R + T) + \frac{1}{12}(\mathrm{RT} + \mathrm{FR} + \mathrm{FT}), \\ c_{1011} &:= \frac{1}{3}I + \frac{5}{24}(F + T) + \frac{1}{12}\mathrm{FT} + \frac{1}{24}(L + R) \\ &\quad + \frac{1}{48}(\mathrm{LT} + \mathrm{RT} + \mathrm{FL} + \mathrm{FR}), \end{aligned} \tag{3.6}$$

$$\begin{aligned} c_{1110} &:= \frac{1}{3}I + \frac{5}{24}F + \frac{1}{8}(L + T) + \frac{5}{96}(\mathrm{FL} + \mathrm{FT}) \\ &\quad + \frac{1}{48}(D + R + \mathrm{LT}) + \frac{1}{96}(\mathrm{FD} + \mathrm{LD} + \mathrm{RT} + \mathrm{FR}), \\ c_{1101} &:= \frac{1}{3}I + \frac{5}{24}F + \frac{1}{8}(R + T) + \frac{5}{96}(\mathrm{FR} + \mathrm{FT}) \\ &\quad + \frac{1}{48}(D + L + \mathrm{RT}) + \frac{1}{96}(\mathrm{FD} + \mathrm{LT} + \mathrm{RD} + \mathrm{FL}), \\ c_{1200} &:= \frac{1}{3}I + \frac{5}{24}F + \frac{7}{96}(L + R + T + D) \\ &\quad + \frac{1}{32}(\mathrm{FL} + \mathrm{FR} + \mathrm{FT} + \mathrm{FD}) + \frac{1}{96}(\mathrm{RT} + \mathrm{RD} + \mathrm{LT} + \mathrm{LD}). \end{aligned} \tag{3.7}$$

For the BB-coefficients of $s_f|_T$ associated with points in $\mathcal{L}_2$ (see Fig. 9), we set

$$\begin{aligned} c_{2010} &:= \frac{3}{8}I + \frac{7}{48}(F + T + L) + \frac{1}{48}(R + D + B + \mathrm{LT} + \mathrm{FL} + \mathrm{FT}) \\ &\quad + \frac{1}{96}(\mathrm{RT} + \mathrm{BT} + \mathrm{FR} + \mathrm{FD} + \mathrm{LD} + \mathrm{BL}), \\ c_{2001} &:= \frac{3}{8}I + \frac{7}{48}(F + T + R) + \frac{1}{48}(L + D + B + \mathrm{RT} + \mathrm{FR} + \mathrm{FT}) \\ &\quad + \frac{1}{96}(\mathrm{LT} + \mathrm{BT} + \mathrm{FL} + \mathrm{FD} + \mathrm{RD} + \mathrm{BR}), \\ c_{2100} &:= \frac{3}{8}I + \frac{1}{12}(T + R + L + D) + \frac{1}{64}(\mathrm{FT} + \mathrm{FR} + \mathrm{FL} + \mathrm{FD}) \\ &\quad + \frac{7}{48}F + \frac{1}{48}B + \frac{1}{96}(\mathrm{RT} + \mathrm{LD} + \mathrm{LT} + \mathrm{RD}) \\ &\quad + \frac{1}{192}(\mathrm{BT} + \mathrm{BR} + \mathrm{BL} + \mathrm{BD}). \end{aligned} \tag{3.8}$$

Finally, for the BB-coefficient of $s_f|_T$ associated with the point ξ_{3000} in $\mathcal{L}_3$, we set

$$c_{3000} := \frac{3}{8}I + \frac{1}{12}(T + F + L + R + D + B) + \frac{1}{96}(\mathrm{LT} + \mathrm{FL} + \mathrm{FT} + \mathrm{RT} + \mathrm{BT} + \mathrm{FR} + \mathrm{FD} + \mathrm{LD} + \mathrm{BD} + \mathrm{BR} + \mathrm{RD} + \mathrm{BL}). \tag{3.9}$$

Note that in (3.3)–(3.9), the BB-coefficients are set locally. More precisely, for any tetrahedron T in $\triangle$, we only use data values at the centres of Ω_T, which is the union of the boxes intersecting the box Q_T containing T. In the following sections, we show that the above setting of the BB-coefficients is done very carefully, so that the spline s_f resulting from (3.3)–(3.9) is globally C^1 smooth, and satisfies certain natural approximation properties.

4. Smoothness properties of the quasi-interpolation operator

In this section, we analyse smoothness properties satisfied by the cubic spline s_f resulting from the method described in Section 3.

First, we note that it can be easily checked that for each pair of tetrahedra in $\triangle$ sharing a common triangular face, each BB-coefficient of s_f associated with a domain point located in that face is uniquely determined: the symmetry of the formulae in (3.3) makes the value of the BB-coefficient independent of the choice of the adjacent tetrahedra. Hence, the spline s_f is a continuous function on Ω, and thus $s_f \in \mathcal{S}_0$. On the other hand, it is non-trivial to see that we set the BB-coefficients of s_f so that the spline s_f is a C^1 function on Ω. The next theorem shows that this is indeed the case.

THEOREM 4.1 The cubic quasi-interpolating spline s_f is in $C^1(\Omega)$, or, equivalently, the quasi-interpolation operator $\mathcal{Q}$: $C(\Omega) \to \mathcal{S}_0$ defined by

$$\mathcal{Q}(f) := s_f, \qquad \text{for each } f \in C(\Omega), \tag{4.1}$$

is a linear operator mapping into the spline space $\mathcal{S}$ defined in (2.1).

Proof. We have to check that the C^1 smoothness conditions across all the faces of each tetrahedron T in $\triangle$ are satisfied. Without loss of generality, let $T = \langle v_0, v_1, v_2, v_3 \rangle$ be the same tetrahedron as in Section 3, with the vertices $v_0 = v_{ijk}$, $v_1 = (v_{ijk} + v_{i-1jk})/2$, $v_2 = (v_{ijk} + v_{i-1j-1k+1})/2$ and $v_3 = (v_{ijk} + v_{i-1j+1k+1})/2$. We have to show that all C^1smoothness conditions in (2.8)–(2.9) are simultaneously satisfied. Since verifying those conditions involves nothing but elementary computations, we only give here a verification for some of them.

We consider the smoothness across $F = \langle v_0, v_1, v_2 \rangle$. Let $\tilde{T} := \langle v_0, v_1, v_2, \tilde{v}_3 \rangle$ be the tetrahedron sharing F with T, i.e. $\tilde{v}_3 = (v_{ijk} + v_{i-1j-1k-1})/2$. Applying the symmetric version of the third equation in (3.3) to set the BB-coefficient $\tilde{c}_{0021}$ of s_f, we obtain

$$\tilde{c}_{0021} = \frac{5}{24}(I + F + L + \mathrm{FL}) + \frac{1}{24}(T + \mathrm{FT} + \mathrm{LT} + \mathrm{FLT}).$$

Hence, we get from (3.3)

$$c_{0021} + \tilde{c}_{0021} = \frac{5}{24}(I + F + T + \mathrm{FT}) + \frac{1}{24}(L + \mathrm{FL} + \mathrm{LT} + \mathrm{FLT}) + \frac{5}{24}(I + F + L + \mathrm{FL}) + \frac{1}{24}(T + \mathrm{FT} + \mathrm{LT} + \mathrm{FLT})$$

$$= \frac{5}{12}(I + F) + \frac{1}{4}(T + \mathrm{FT} + L + \mathrm{FL}) + \frac{1}{12}(\mathrm{LT} + \mathrm{FLT})$$
$$= 2c_{0120},$$

where in the last equality, we use the value of c_{0120} from (3.4). This shows that the first smoothness condition in (2.6) is satisfied. All the remaining conditions in (2.6), and the conditions in (2.7), (2.9) can be checked similarly.

Next we consider the smoothness across the face $F = \langle v_0, v_2, v_3\rangle$. Let $\tilde{T} := \langle v_0, \tilde{v}_1, v_2, v_3\rangle$ be the tetrahedron sharing F with T, i.e. $\tilde{v}_1 = (v_{ijk} + v_{ijk+1})/2$. Applying the symmetric version of the third equation in (3.8) to set the BB-coefficient $\tilde{c}_{2100}$ of s_f, we obtain

$$\begin{aligned}\tilde{c}_{2100} = {} & \frac{3}{8}I + \frac{1}{12}(F + R + L + B) + \frac{1}{64}(\mathrm{FT} + \mathrm{RT} + \mathrm{LT} + \mathrm{BT}) \\ & + \frac{7}{48}T + \frac{1}{48}D + \frac{1}{96}(\mathrm{FR} + \mathrm{FL} + \mathrm{BR} + \mathrm{BL}) \\ & + \frac{1}{192}(\mathrm{FD} + \mathrm{RD} + \mathrm{LD} + \mathrm{BD}).\end{aligned}$$

Now it follows from the formulae for the BB-coefficients c_{2100}, c_{2010} and c_{2001} in (3.8) that

$$\begin{aligned}& c_{2100} + \tilde{c}_{2100} - (c_{2010} + c_{2001})/2 \\ & = \Bigg(\frac{3}{8}I + \frac{1}{12}(T + R + L + D) + \frac{1}{64}(\mathrm{FT} + \mathrm{FR} + \mathrm{FL} + \mathrm{FD}) + \frac{7}{48}F \\ & \quad + \frac{1}{48}B + \frac{1}{96}(\mathrm{RT} + \mathrm{LD} + \mathrm{LT} + \mathrm{RD}) + \frac{1}{192}(\mathrm{BT} + \mathrm{BR} + \mathrm{BL} + \mathrm{BD})\Bigg) \\ & \quad + \Bigg(\frac{3}{8}I + \frac{1}{12}(F + R + L + B) + \frac{1}{64}(\mathrm{FT} + \mathrm{RT} + \mathrm{LT} + \mathrm{BT}) + \frac{7}{48}T \\ & \quad + \frac{1}{48}D + \frac{1}{96}(\mathrm{FR} + \mathrm{FL} + \mathrm{BR} + \mathrm{BL}) + \frac{1}{192}(\mathrm{FD} + \mathrm{RD} + \mathrm{LD} + \mathrm{BD})\Bigg) \\ & \quad - \Bigg[\Bigg(\frac{3}{8}I + \frac{7}{48}(F + T + L) + \frac{1}{48}(R + D + B + \mathrm{LT} + \mathrm{FL} + \mathrm{FT}) \\ & \quad + \frac{1}{96}(\mathrm{RT} + \mathrm{BT} + \mathrm{FR} + \mathrm{FD} + \mathrm{LD} + \mathrm{BL})\Bigg) \\ & \quad + \Bigg(\frac{3}{8}I + \frac{7}{48}(F + T + R) + \frac{1}{48}(L + D + B + \mathrm{RT} + \mathrm{FR} + \mathrm{FT}) \\ & \quad + \frac{1}{96}(\mathrm{LT} + \mathrm{BT} + \mathrm{FL} + \mathrm{FD} + \mathrm{RD} + \mathrm{BR})\Bigg)\Bigg]\Bigg/2.\end{aligned}$$

Some elementary computations show that

$$\begin{aligned}
&c_{2100} + \tilde{c}_{2100} - (c_{2010} + c_{2001})/2 \\
&\quad = \frac{3}{8} I + \frac{1}{12}(T + R + L + D + F + B) + \frac{1}{96}(\mathrm{FT} + \mathrm{FR} + \mathrm{FL} \\
&\qquad + \mathrm{FD} + \mathrm{RT} + \mathrm{LD} + \mathrm{LT} + \mathrm{RD} + \mathrm{BT} + \mathrm{BR} + \mathrm{BL} + \mathrm{BD}).
\end{aligned}$$

Comparing this result with the formula for the BB-coefficient c_{3000} in (3.9), we see that the first smoothness condition in (2.8) is satisfied. All the remaining smoothness conditions in (2.8) can be verified similarly. The proof of the theorem is complete. □

5. Properties of the quasi-interpolation operator

In this section, we summarize some important properties of the quasi-interpolation operator $\mathcal{Q}$ defined in (4.1). These results are used for deriving the error bounds in Section 6.

We begin with certain reproduction properties of the quasi-interpolation operator. The next lemma shows that $\mathcal{Q}$ reproduces trilinear polynomials.

LEMMA 5.1 For any p in $\mathcal{T}_3 := \mathrm{span}\{1, x, y, z, xy, xz, yz, xyz\} \subseteq \mathcal{P}_3$, we have $\mathcal{Q}(p) = p$.

Proof. Due to the symmetry of our scheme in Section 3, it suffices to show that the monomials from $\{1, x, xy, xyz\}$ are reproduced by $\mathcal{Q}$. This is clear for the first monomial in this set, because our choice (as in (3.3)–(3.9)) of the weights $\omega_{i_0 j_0 k_0}$ in (3.2) yields

$$\sum_{i_0, j_0, k_0 \in \{-1,0,1\}} \omega_{i_0 j_0 k_0} = 1,$$

and, therefore, each BB-coefficient of the spline $\mathcal{Q}(1)$ has the value 1, assuring that $\mathcal{Q}(1) = 1$. The reproduction of any other monomial q can be checked directly by comparing the BB-coefficients of the spline $\mathcal{Q}(q)$ on an arbitrary tetrahedron in $\triangle$ with the values of the BB-coefficients of q on this tetrahedron. It suffices to consider the same tetrahedron $T = \langle v_0, v_1, v_2, v_3 \rangle$ as in Section 3 with the vertices $v_0 = v_{ijk}$, $v_1 = (v_{ijk} + v_{i-1jk})/2$, $v_2 = (v_{ijk} + v_{i-1j-1k+1})/2$ and $v_3 = (v_{ijk} + v_{i-1j+1k+1})/2$.

Let $q(x, y) = x$. Its BB-coefficients $c_{ijk\ell}$ with respect to T are

$$\begin{aligned}
&c_{0030} = c_{0003} = c_{0021} = c_{0012} = c_{0120} = c_{0102} \\
&\qquad = c_{0111} = c_{0201} = c_{0300} = \left(i - \frac{1}{2}\right) h, \\
&c_{1020} = c_{1002} = c_{1011} = c_{1110} = c_{1101} = c_{1200} = \left(i - \frac{1}{3}\right) h, \\
&c_{2010} = c_{2001} = c_{2100} = \left(i - \frac{1}{6}\right) h, \quad c_{3000} = ih.
\end{aligned} \tag{5.1}$$

On the other hand, for the monomial $q(x, y) = x$, we have

$$\begin{aligned}
F &= \mathrm{FL} = \mathrm{FR} = \mathrm{FD} = \mathrm{FT} = \mathrm{FLD} = \mathrm{FRD} = \mathrm{FLT} = \mathrm{FRT} = (i-1)h, \\
I &= L = R = T = D = \mathrm{LD} = \mathrm{RD} = \mathrm{LT} = \mathrm{RT} = ih, \\
B &= \mathrm{BL} = \mathrm{BR} = \mathrm{BD} = \mathrm{BT} = \mathrm{BLD} = \mathrm{BRD} = \mathrm{BLT} = \mathrm{BRT} = (i+1)h,
\end{aligned}$$

and from our scheme in (3.3)–(3.9) it follows that

$$
\begin{aligned}
c_{0030} &= \frac{1}{8}(4i + 4(i-1))h = c_{0003},\\
c_{0021} &= \left(\frac{5}{24} + \frac{1}{24}\right)(2i + 2(i-1))h = c_{0012},\\
c_{0120} &= \left(\left(\frac{5}{24} + \frac{1}{24}\right)(2i-1) + \frac{1}{8}(2i + 2(i-1))\right)h = c_{0102},\\
c_{0111} &= \left(\left(\frac{13}{48} + \frac{7}{48}\right)(2i-1) + \left(\frac{1}{32} + \frac{1}{96}\right)(2i + 2(i-1))\right)h,\\
c_{0210} &= \left(\left(\frac{13}{48} + \frac{1}{96}\right)(2i-1) + \left(\frac{17}{192} + \frac{1}{64} + \frac{1}{192}\right)(2i + 2(i-1))\right)h = c_{0201},\\
c_{0300} &= \left(\frac{13}{48}(2i-1) + \left(\frac{5}{96} + \frac{1}{192}\right)(4i + 4(i-1))\right)h,\\
c_{1020} &= \left(\frac{1}{4}i + \frac{1}{6}((i-1) + 2i) + \frac{1}{12}(i + 2(i-1))\right)h = c_{1002},\\
c_{1011} &= \left(\frac{1}{3}i + \frac{5}{24}((i-1) + i) + \frac{1}{12}(i-1) + \frac{1}{24}(2i) + \frac{1}{48}(2i + 2(i-1))\right)h,\\
c_{1110} &= \left(\frac{1}{3}i + \frac{5}{24}(i-1) + \frac{1}{8}(2i) + \frac{5}{96}(2(i-1)) + \frac{1}{48}(3i)\right.\\
&\qquad \left. + \frac{1}{96}(2i + 2(i-1))\right)h = c_{1101},\\
c_{1200} &= \left(\frac{1}{3}i + \frac{5}{24}(i-1) + \left(\frac{7}{96} + \frac{1}{96}\right)(4i) + \frac{1}{32}(4(i-1))\right)h,\\
c_{2010} &= \left(\frac{3}{8}i + \frac{7}{48}(2i + (i-1)) + \frac{1}{48}((i+1) + 3i + 2(i-1))\right.\\
&\qquad \left. + \frac{1}{96}(2(i+1) + 2i + 2(i-1))\right)h = c_{2001},\\
c_{2100} &= \left(\frac{3}{8}i + \left(\frac{1}{12} + \frac{1}{96}\right)(4i) + \frac{1}{64}(4(i-1)) + \frac{7}{48}(i-1)\right.\\
&\qquad \left. + \frac{1}{48}(i+1) + \frac{1}{192}(4(i+1))\right)h,\\
c_{3000} &= \left(\frac{3}{8}i + \frac{1}{12}((i+1) + 4i + (i-1)\right) + \frac{1}{96}(4(i+1) + 4i + 4(i-1)))h.
\end{aligned}
$$

Some elementary calculations now show that these are in fact the BB-coefficients of q with respect to T, as in (5.1). This completes the proof that $\mathcal{Q}(q) = q$. The reproduction of the remaining monomials xy and xyz can be shown similarly, and hence the proof is complete. □

The next lemma shows that certain cubic polynomials are nearly reproduced by our quasi-interpolation operator $\mathcal{Q}$.

LEMMA 5.2 For the cubic polynomials $p \in \mathcal{P}_3$ of the form

$$p(x, y) = \tilde{p}(x, y) + ax^2 + by^2 + cz^2, \quad (x, y) \in \Omega, \tag{5.2}$$

where $\tilde{p} \in \mathcal{T}_3$, we have $\mathcal{Q}(p) = p + \frac{1}{4}(a + b + c)h^2$.

Proof. Lemma 5.1, the linearity of $\mathcal{Q}$ and the symmetry of our quasi-interpolation scheme imply that it suffices to show that

$$\mathcal{Q}(p) = p + \frac{1}{4}h^2, \quad \text{where } p(x, y) = x^2.$$

To this end, we basically use the same method as in the proof of Lemma 5.1, namely, we compare the BB-coefficients of the spline $\mathcal{Q}(p)$ on the same tetrahedron $T := \langle v_0, v_, v_2, v_3 \rangle$ in Q_{ijk} as in Lemma 5.1 with the values of the BB-coefficients of p on this tetrahedron. We begin by computing the values $p_{ijk\ell} := p(\xi_{ijk\ell})$ in T:

$$\begin{aligned} p_{0jk\ell} &= \left(i - \frac{1}{2}\right)^2 h^2, \quad j + k + \ell = 3, \quad p_{1jk\ell} = \left(i - \frac{1}{3}\right)^2 h^2, \quad j + k + \ell = 2, \\ p_{2jk\ell} &= \left(i - \frac{1}{6}\right)^2 h^2, \quad j + k + \ell = 1, \quad p_{3000} = i^2 h^2. \end{aligned} \tag{5.3}$$

Using these values of p and (2.4), we compute the BB-coefficients $\tilde{c}_{ijk\ell}$ of p on T:

$$\begin{aligned} \tilde{c}_{0jk\ell} &= \left(i - \frac{1}{2}\right)^2 h^2, \quad j + k + \ell = 3, \\ \tilde{c}_{1jk\ell} &= \left(i^2 - \frac{2}{3}i + \frac{1}{12}\right) h^2, \quad j + k + \ell = 2, \\ \tilde{c}_{2jk\ell} &= \left(i^2 - \frac{1}{3}i\right) h^2, \quad j + k + \ell = 1, \quad \tilde{c}_{3000} = i^2 h^2. \end{aligned} \tag{5.4}$$

On the other hand, for $p(x, y) = x^2$, we have

$$F = \mathrm{FL} = \mathrm{FR} = \mathrm{FD} = \mathrm{FT} = \mathrm{FLD} = \mathrm{FRD} = \mathrm{FLT} = \mathrm{FRT} = (i - 1)^2 h^2,$$

$$I = L = R = T = D = \mathrm{LD} = \mathrm{RD} = \mathrm{LT} = \mathrm{RT} = i^2 h^2,$$

$$B = \mathrm{BL} = \mathrm{BR} = \mathrm{BD} = \mathrm{BT} = \mathrm{BLD} = \mathrm{BRD} = \mathrm{BLT} = \mathrm{BRT} = (i + 1)^2 h^2,$$

and our quasi-interpolation scheme implies that

$$c_{0030} = c_{0003} = \frac{1}{8}(4i^2 + 4(i-1)^2)h^2 = \left(\left(i - \frac{1}{2}\right)^2 + \frac{1}{4}\right)h^2,$$

$$c_{0021} = c_{0012} = \left(\frac{5}{24} + \frac{1}{24}\right)(2i^2 + 2(i-1)^2)h^2 = \left(\left(i - \frac{1}{2}\right)^2 + \frac{1}{4}\right)h^2,$$

$$c_{0012} = c_{0120} = \left(\left(\frac{5}{24} + \frac{1}{24}\right)(i^2 + (i-1)^2) + \frac{1}{8}(2i^2 + 2(i-1)^2)\right)h^2$$

$$= \left(\left(i - \frac{1}{2}\right)^2 + \frac{1}{4}\right)h^2,$$

$$c_{0111} = \left(\left(\frac{13}{48} + \frac{7}{48}\right)(i^2 + (i-1)^2) + \left(\frac{1}{32} + \frac{1}{96}\right)(2i^2 + 2(i-1)^2)\right)h^2$$

$$= \left(\left(i - \frac{1}{2}\right)^2 + \frac{1}{4}\right)h^2,$$

$$c_{0210} = c_{0201} = \left(\left(\frac{13}{48} + \frac{1}{96}\right)(i^2 + (i-1)^2\right)$$

$$+ \left(\frac{17}{192} + \frac{1}{64} + \frac{1}{192}\right)(2i^2 + 2(i-1)^2))h^2$$

$$= \left(\left(i - \frac{1}{2}\right)^2 + \frac{1}{4}\right)h^2,$$

$$c_{0300} = \left(\frac{13}{48}\left(i^2 + (i-1)^2\right) + \left(\frac{5}{96} + \frac{1}{192}\right)(4i^2 + 4(i-1)^2)\right)h^2$$

$$= \left(\left(i - \frac{1}{2}\right)^2 + \frac{1}{4}\right)h^2,$$

$$c_{1020} = c_{1002} = \left(\frac{1}{4}i^2 + \frac{1}{6}((i-1)^2 + 2i^2) + \frac{1}{12}(i^2 + 2(i-1)^2)\right)h^2$$

$$= \left(\left(i^2 - \frac{2}{3}i + \frac{1}{12}\right) + \frac{1}{4}\right)h^2,$$

$$c_{1011} = \left(\frac{1}{3}i^2 + \frac{5}{24}((i-1)^2 + i^2) + \frac{1}{12}(i-1)^2 + \frac{1}{24}(2i^2) + \frac{1}{48}(2i^2 + 2(i-1)^2)\right)h^2$$

$$= \left(\left(i^2 - \frac{2}{3}i + \frac{1}{12}\right) + \frac{1}{4}\right)h^2,$$

$$
\begin{aligned}
c_{1110} = c_{1101} &= \left(\left(\frac{1}{3}+\frac{2}{8}+\frac{3}{48}\right)i^2 + \left(\frac{5}{24}+\frac{10}{96}\right)(i-1)^2 + \frac{2}{96}(i^2+(i-1)^2)\right)h^2 \\
&= \left(\left(i^2 - \frac{2}{3}i + \frac{1}{12}\right) + \frac{1}{4}\right)h^2, \\
c_{1200} &= \left(\left(\frac{1}{3}+\frac{28}{96}+\frac{4}{96}\right)i^2 + \left(\frac{5}{24}+\frac{4}{32}\right)(i-1)^2\right)h^2 \\
&= \left(\left(i^2 - \frac{2}{3}i + \frac{1}{12}\right) + \frac{1}{4}\right)h^2, \\
c_{2010} = c_{2001} &= \left(\frac{3}{8}i^2 + \frac{7}{48}(2i^2+(i-1)^2) + \frac{1}{48}((i+1)^2+3i^2\right. \\
&\qquad \left. + 2(i-1)^2) + \frac{1}{96}(2(i+1)^2+2i^2+2(i-1)^2)\right)h^2 \\
&= \left(\left(i^2 - \frac{1}{3}i\right) + \frac{1}{4}\right)h^2, \\
c_{2100} &= \left(\left(\frac{3}{8}+\frac{4}{12}+\frac{4}{96}\right)i^2 + \left(\frac{4}{64}+\frac{7}{48}\right)(i-1)^2 + \left(\frac{1}{48}+\frac{4}{192}\right)(i+1)^2\right)h^2 \\
&= \left(\left(i^2 - \frac{1}{3}i\right) + \frac{1}{4}\right)h^2, \\
c_{3000} &= \left(\frac{3}{8}i^2 + \frac{1}{12}((i+1)^2+4i^2+(i-1)^2) + \frac{1}{96}(4(i+1)^2+4i^2+4(i-1)^2)\right)h^2 \\
&= \left(i^2 + \frac{1}{4}\right)h^2.
\end{aligned}
$$

It is now clear that $\tilde{c}_{ijk\ell} + \frac{1}{4}h^2 = c_{ijk\ell}$, for all $i+j+k+\ell = 3$, and hence, the proof of the lemma is complete. □

The next corollary is a direct consequence of the previous lemma. From here on, we denote by $D_x^\alpha D_y^\beta D_z^\gamma$ higher-order partial derivatives.

COROLLARY 5.3 For the cubic polynomials $p \in \mathcal{P}_3$ of the form

$$p(x, y) = \tilde{p}(x, y) + ax^2 + by^2 + cz^2, \quad (x, y) \in \Omega,$$

where $\tilde{p} \in \mathcal{T}_3$, we have for all $\alpha + \beta + \gamma \in \{1, 2, 3\}$

$$D_x^\alpha D_y^\beta D_z^\gamma \mathcal{Q}(p) = \mathcal{Q}(D_x^\alpha D_y^\beta D_z^\gamma p).$$

Proof. From Lemma 5.2, it follows that $Q(p) = p + \kappa$, where κ is a constant. This implies that for $\alpha + \beta + \gamma \in \{1, 2, 3\}$,

$$D_x^\alpha D_y^\beta D_z^\gamma \mathcal{Q}(p) = D_x^\alpha D_y^\beta D_z^\gamma (p + \kappa) = D_x^\alpha D_y^\beta D_z^\gamma p.$$

Moreover, for $\alpha+\beta+\gamma \in \{1, 2, 3\}$, we have $D_x^\alpha D_y^\beta D_z^\gamma\, p \in \mathcal{T}_3$, and, therefore, Lemma 5.1 implies that

$$D_x^\alpha D_y^\beta D_z^\gamma\, p = \mathcal{Q}(D_x^\alpha D_y^\beta D_z^\gamma\, p).$$

The proof is complete. □

Given a compact set $B \subseteq \tilde{\Omega} =: [0, (n+1)h] \times [0, (m+1)h] \times [(r+1)h]$, for any $f \in C(\tilde{\Omega})$, we denote by

$$\|f\|_B := \sup\{|f(x, y)|: (x, y) \in B\}, \tag{5.5}$$

the uniform norm of f. The next theorem shows that the operator $\mathcal{Q}$ is uniformly bounded. In particular, the result implies that the computation of the splines can be done in a very stable way, because the associated (global) operator norm $\|\mathcal{Q}\|$ of $\mathcal{Q}$ satisfies

$$\|\mathcal{Q}\| := \sup\{\|\mathcal{Q}(f)\|_\Omega: \|f\|_{\tilde{\Omega}} = 1\} = 1.$$

THEOREM 5.4 For any tetrahedron T in $\triangle$ contained in the box Q_T in $\diamondsuit$,

$$\|\mathcal{Q}(f)\|_T \leqslant \|f\|_{\Omega_T},$$

where Ω_T is the union of the boxes intersecting Q_T.

Proof. According to our method, all the BB-coefficients $c_{ijk\ell}$ of the polynomial piece $p = s_f|_T = \mathcal{Q}(f)|_T$ on T in its BB-form (2.2) are determined by using the values of f at the points from the boxes intersecting Q_T. Since the weights $\omega_{i_0 j_0 k_0}$ in (3.2) are non-negative and sum up to 1, it follows from (3.3)–(3.9) that

$$|c_{ijk\ell}| \leqslant \|f\|_{\Omega_T}, \quad i+j+k+\ell = 3.$$

The assertion now follows, since from (2.3) we obtain

$$\|p\|_T \leqslant \max\{|c_{ijk\ell}|: i+j+k+\ell = 3\} \leqslant \|f\|_{\Omega_T}.$$

The proof of the theorem is complete. □

6. Approximation properties of the quasi-interpolating spline

In this section, we give error bounds for the quasi-interpolating spline as well as for its derivatives. We first prove that—as in the case of the trilinear model mentioned in Section 1—the quasi-interpolating spline $s_f := \mathcal{Q}(f)$ approximates twice differentiable functions with order two, while their first derivatives are simultaneously approximated with order one. In addition, we show that, if the data come from a three times differentiable function, then the derivatives of s_f have an advantageous behaviour, namely, they approximate the derivatives $D_x^\alpha D_y^\beta D_z^\gamma f$, $\alpha+\beta+\gamma = 1, 2, 3$, with nearly optimal approximation order. This is a non-standard phenomenon because it means that for this function class, the quasi-interpolating spline and its first derivatives yield the same approximation order.

We begin by giving a (local) error bound for $f - \mathcal{Q}(f)$ and its first derivatives in the uniform norm for the case when f is two times continuously differentiable. Here and in the following, we let

$$\|D^r f\|_B := \max\{\|D_x^\alpha D_y^\beta f D_z^\gamma f\|_B: \alpha+\beta+\gamma = r\}$$

for any (piecewise) r-times differentiable function f, and B as in (5.5).

THEOREM 6.1 Let T, Q_T, Ω_T be as in Theorem 5.4, and $f \in C^2(\Omega_T)$. Then,

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - \mathcal{Q}(f))\|_T \leqslant K_0 \|D^2 f\|_{\Omega_T} h^{2-\alpha-\beta-\gamma}, \quad \alpha+\beta+\gamma = 0, 1, 2, \tag{6.1}$$

where $K_0 > 0$ is an absolute constant independent of f and h.

Proof. We consider the Lagrange form of the remainder term in the Taylor expansion of f at the centre of Ω_T and obtain $p_f \in \mathcal{P}_1 := \text{span}\{1, x, y, z\}$ with the property

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - p_f)\|_{\Omega_T} \leqslant C_0 \|D^2 f\|_{\Omega_T} h^{2-\alpha-\beta-\gamma}, \quad \alpha+\beta+\gamma = 0, 1, 2, \tag{6.2}$$

where C_0 is an absolute constant independent of f and h. Since $D_x^\alpha D_y^\beta D_z^\gamma p_f \in \mathcal{T}_3$, Lemma 5.1 implies that

$$\mathcal{Q}(D_x^\alpha D_y^\beta D_z^\gamma p_f) = D_x^\alpha D_y^\beta D_z^\gamma p_f, \quad \alpha+\beta+\gamma = 0, 1, 2.$$

Therefore,

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - \mathcal{Q}(f))\|_T \leqslant \|D_x^\alpha D_y^\beta D_z^\gamma (f - p_f)\|_{\Omega_T} + \|D_x^\alpha D_y^\beta D_z^\gamma \mathcal{Q}(f - p_f)\|_T,$$

for all $\alpha+\beta+\gamma = 0, 1, 2$. In view of (6.2), it now suffices to estimate the second term of these inequalities. For $\alpha+\beta+\gamma = 0$, we use Theorem 5.4 and the error bound (6.2) to obtain

$$\|\mathcal{Q}(f - p_f)\|_T \leqslant \|f - p_f\|_{\Omega_T} \leqslant C_0 \|D^2 f\|_{\Omega_T} h^2.$$

For $\alpha+\beta+\gamma = 1, 2$, we first use a Markov type inequality as in Nürnberger *et al.* (2005a) with a constant $M_0 \geqslant 1$, and then we use Theorem 5.4 to obtain

$$\|D_x^\alpha D_y^\beta D_z^\gamma \mathcal{Q}(f - p_f)\|_T \leqslant M_0 h^{-\alpha-\beta-\gamma} \|\mathcal{Q}(f - p_f)\|_T \leqslant M_0 C_0 \|D^2 f\|_{\Omega_T} h^{2-\alpha-\beta-\gamma}.$$

Combining these inequalities leads to (6.1) with the constant $K_0 = (1 + M_0)C_0$ independent of f and h. The proof is complete. □

Next we provide a (local) error bound for $f - \mathcal{Q}(f)$ and its derivatives in the uniform norm for the case when f is three times continuously differentiable.

THEOREM 6.2 Let T, Q_T, Ω_T be as in Theorem 5.4, and $f \in C^3(\Omega_T)$. Then,

$$\|f - \mathcal{Q}(f)\|_T \leqslant K_1 \|D^2 f\|_{\Omega_T} h^2 + K_2 \|D^3 f\|_{\Omega_T} h^3, \tag{6.3}$$

and for all $\alpha+\beta+\gamma = 1, 2, 3$,

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - \mathcal{Q}(f))\|_T \leqslant K_3 \|D^3 f\|_{\Omega_T} h^{3-\alpha-\beta-\gamma},$$

where $K_1, K_2, K_3 > 0$ are absolute constants independent of f and h.

Proof. First, the Taylor expansion of f at the centre of Ω_T with the Lagrange form of the remainder term shows the existence of $p_f \in \mathcal{P}_2$ with

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - p_f)\|_{\Omega_T} \leqslant C_1 \|D^3 f\|_{\Omega_T} h^{3-\alpha-\beta-\gamma}, \quad \alpha+\beta+\gamma = 0, \ldots, 3, \tag{6.4}$$

where C_1 is an absolute constant independent of f and h. The triangle inequality and (6.4) yield

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - \mathcal{Q}(f))\|_T \leqslant C_1 \|D^3 f\|_{\Omega_T} h^{3-\alpha-\beta-\gamma} + \|D_x^\alpha D_y^\beta D_z^\gamma (\mathcal{Q}(f) - p_f)\|_T,$$

for all $\alpha+\beta+\gamma = 0, \dots, 3$. Hence, we have to estimate the second term. We first consider $\alpha+\beta+\gamma = 0$. Obviously, $p_f \in \mathcal{P}_2$ can be written in the form (5.2), and hence it follows from Lemma 5.2 that

$$\|\mathcal{Q}(f) - p_f\|_T \leqslant \|\mathcal{Q}(f - p_f)\|_T + \frac{1}{8}\max\{\|D_x^2 f\|_T, \|D_y^2 f\|_T, \|D_z^2 f\|_T\}h^2.$$

Theorem 5.4 and the error bound (6.4) imply that

$$\|\mathcal{Q}(f - p_f)\|_T \leqslant \|f - p_f\|_{\Omega_T} \leqslant C_1\|D^3 f\|_{\Omega_T} h^3,$$

and we obtain the first estimate in (6.3) with constants $K_2 = 2C_1$ and $K_1 = \frac{1}{8}$, respectively. Next we consider $\alpha+\beta+\gamma = 1, 2, 3$. Since the derivatives of p_f are contained in $\mathcal{T}_3$, we obtain from Lemma 5.1

$$D_x^\alpha D_y^\beta D_z^\gamma (\mathcal{Q}(f) - p_f) = D_x^\alpha D_y^\beta D_z^\gamma \mathcal{Q}(f) - \mathcal{Q}(D_x^\alpha D_y^\beta D_z^\gamma p_f).$$

Corollary 5.3 now implies that

$$D_x^\alpha D_y^\beta D_z^\gamma (\mathcal{Q}(f) - p_f) = D_x^\alpha D_y^\beta D_z^\gamma (\mathcal{Q}(f) - \mathcal{Q}(p_f)) = D_x^\alpha D_y^\beta D_z^\gamma \mathcal{Q}(f - p_f),$$

for all $\alpha + \beta + \gamma = 1, 2, 3$. Applying a Markov type inequality as in Nürnberger *et al.* (2005a) with a constant $M_1 \geqslant 1$, and Theorem 5.4, we obtain

$$\begin{aligned}\|D_x^\alpha D_y^\beta D_z^\gamma \mathcal{Q}(f - p_f)\|_T &\leqslant M_1 h^{-\alpha-\beta-\gamma}\|\mathcal{Q}(f - p_f)\|_T \\ &\leqslant M_1 h^{-\alpha-\beta-\gamma}\|f - p_f\|_{\Omega_T} \\ &\leqslant M_1 C_1\|D^3 f\|_{\Omega_T} h^{3-\alpha-\beta-\gamma}\end{aligned}$$

for all $\alpha + \beta + \gamma = 1, 2, 3$. Combining these inequalities leads to the second inequality in (6.3) with constant $K_3 = (1 + M_1)C_1$ independent of f and h. The proof is complete. □

We conclude this section with a summary of global error bounds. The results follow from Theorems 6.1 and 6.2, respectively.

THEOREM 6.3

(i) If $f \in C^2(\tilde{\Omega})$, then

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - \mathcal{Q}(f))\|_\Omega \leqslant K_0\|D^2 f\|_{\tilde{\Omega}} h^{2-\alpha-\beta-\gamma}, \quad \alpha + \beta + \gamma = 0, 1, 2, \tag{6.5}$$

where $K_0 > 0$ is the constant from Theorem 6.1.

(ii) If $f \in C^3(\tilde{\Omega})$, then

$$\|f - \mathcal{Q}(f)\|_\Omega \leqslant K_1\|D^2 f\|_{\tilde{\Omega}} h^2 + K_2\|D^3 f\|_{\tilde{\Omega}} h^3, \tag{6.6}$$

and for all $\alpha + \beta + \gamma = 1, \dots, 3$,

$$\|D_x^\alpha D_y^\beta D_z^\gamma (f - \mathcal{Q}(f))\|_\Omega \leqslant K_3\|D^3 f\|_{\tilde{\Omega}} h^{3-\alpha-\beta-\gamma},$$

where $K_1, K_2, K_3 > 0$ are the constants from Theorem 6.2.

7. Numerical tests and remarks

In order to illustrate the approximation properties of the quasi-interpolating splines, we provide two numerical examples based on synthetic data sampled from smooth test functions. Thus, in the tests presented here, we ignore the round off effects, use double-precision computations and assume that all the data values are exact.

We first consider the Marschner–Lobb test function (Marschner & Lobb, 1994) defined as

$$\mathrm{ml}(v) = \frac{2}{5}\left(1 - \sin(\pi z/2) + \frac{1}{4}\left(1 + \cos\left(12\pi \cos\left(\pi\sqrt{x^2 + y^2}/2\right)\right)\right)\right),$$

for all $v = (x, y, z) \in [-1, 1] \times [-1, 1] \times [-1, 1]$, so that $\mathrm{ml}(v) \in [0, 1]$. This function of extreme oscillation is used frequently in the area of volume visualization because it provides a difficult test for any efficient 3D reconstruction method, in particular, in the cases when only very few data samples are taken and simultaneous approximation of derivatives plays an important role.

We compute the cubic quasi-interpolating C^1 splines s_{ml} according to our approach (for $n = m = r$), where we choose decreasing h and consider different kinds of errors. The numerical results are given in Table 1. The first column contains values of $h = 1/n$. The total number N of data points is huge, namely, $N = (n+2)^3$. Thus, the biggest number of data sites we have is about 17×10^6. The computation time for the largest test on a standard machine does not exceed a few seconds. The remaining columns contain different types of errors. We denote by $\mathrm{err}^{\mathrm{ml}}_{\mathrm{data}}$ the maximal error of the function values at the grid points, and $\mathrm{err}^{\mathrm{ml}}_{\mathrm{max}}$ is the maximal error of the spline in the uniform norm on Ω. The latter error is computed approximately on a fine discretization $\mathcal{T}$ of the domain by choosing a fixed (high) number of uniformly distributed points in each tetrahedron of Δ. We compute the values of the test function and its approximating spline within machine precision, and calculate the maximal error on $\mathcal{T}$. For the sake of completeness, we also give the (approximate) average error $\mathrm{err}^{\mathrm{ml}}_{\mathrm{mean}}$, and the (approximate) root mean square error $\mathrm{err}^{\mathrm{ml}}_{\mathrm{rms}}$, where the computations are based on the same sets $\mathcal{T}$.

The results in Table 1 confirm that the quasi-interpolating splines yield approximation order two, since in each row the error decreases by about the factor of four while h goes down to $h/2$. Moreover, we computed the analogous errors for the first derivative $D_x(\mathrm{ml})$ of ml. All the computations were done in the same manner as for the values. As it is well-known in general spline theory, spline operators possess the advantageous property to simultaneously approximate derivatives of a smooth function, even in the case when only the values of this function are used. The results in Table 2 indicate that the corresponding errors behave in the same way as the errors of the values. Therefore, the error of the first derivative D_x is nearly optimal as we proved in Theorem 6.3. Considering general theory, this is a non-standard phenomenon: for a smooth function, the first derivatives of the quasi-interpolating spline provide the same order of accuracy as the spline itself.

TABLE 1 *Approximation of ml by s_{ml}*

$1/h$	$\mathrm{err}^{\mathrm{ml}}_{\mathrm{mean}}$	$\mathrm{err}^{\mathrm{ml}}_{\mathrm{rms}}$	$\mathrm{err}^{\mathrm{ml}}_{\mathrm{max}}$	$\mathrm{err}^{\mathrm{ml}}_{\mathrm{data}}$
16	0.065039	0.078119	0.184461	0.075148
32	0.047294	0.055252	0.122054	0.078329
64	0.017678	0.020732	0.039583	0.034708
128	0.004956	0.005843	0.010533	0.010167
256	0.001276	0.001506	0.002671	0.002648

TABLE 2 *Approximation of $D_x(ml)$ by $D_x(s_{ml})$*

$1/h$	$\mathrm{err}_{\mathrm{mean}}^{D_x(\mathrm{ml})}$	$\mathrm{err}_{\mathrm{rms}}^{D_x(\mathrm{ml})}$	$\mathrm{err}_{\mathrm{max}}^{D_x(\mathrm{ml})}$	$\mathrm{err}_{\mathrm{data}}^{D_x(\mathrm{ml})}$
16	4.1498	5.2147	12.7069	10.1055
32	3.1971	4.1962	12.8251	12.5353
64	1.2138	1.6369	5.6238	5.6195
128	0.3367	0.4578	1.6029	1.5988
256	0.0866	0.1180	0.4141	0.4128

TABLE 3 *Approximation of f by s_f*

$1/h$	$\mathrm{err}_{\mathrm{mean}}^{f}$	$\mathrm{err}_{\mathrm{rms}}^{f}$	$\mathrm{err}_{\mathrm{max}}^{f}$	$\mathrm{err}_{\mathrm{data}}^{f}$
16	0.0035295	0.0061525	0.0426452	0.0426404
32	0.0008831	0.0015573	0.0109651	0.0109638
64	0.0002203	0.0003903	0.0027608	0.0027605
128	0.0000550	0.0000976	0.0006914	0.0006913
256	0.0000137	0.0000244	0.0001729	0.0001729

In the second test, we use the smooth trivariate test function of Franke type

$$
\begin{aligned}
f(v) = {} & \frac{1}{2}\,\mathrm{e}^{-10((x-\frac{1}{4})^2+(y-\frac{1}{4})^2)} + \frac{3}{4}\,\mathrm{e}^{-16((x-\frac{1}{4})^2+(y-\frac{1}{4})^2+(z-\frac{1}{4})^2)} \\
& + \frac{1}{2}\,\mathrm{e}^{-10((x-\frac{3}{4})^2+(y-\frac{1}{8})^2+(z-\frac{1}{2})^2)} - \frac{1}{4}\,\mathrm{e}^{-20((x-\frac{3}{4})^2+(y-\frac{3}{4})^2)},
\end{aligned}
$$

for all $v = (x, y, z) \in [-1/2, 1/2] \times [-1/2, 1/2] \times [-1/2, 1/2]$, so that $f(v) \in [0, 1.3]$. This function is less oscillatory than the Marschner–Lobb function. In Tables 3 and 4, we summarize the results of our computations for this test function, where we use the same notations for different errors as in Table 1. Again, this confirms the results of Theorem 6.3—in particular, that the derivatives of the approximating spline s_f converge to the derivatives of f with the nearly optimal order.

We conclude the paper with some remarks.

REMARK 7.1 With the notation from Section 3, it is straightforward to evaluate the partial derivatives of s_f in x, y and z directions at the corners of the boxes in $\diamondsuit$ and at the points in V:

(i) For each vertex w of $\diamondsuit$, we have

$$
\begin{aligned}
D_x s_f(w) &= \frac{1}{(4h)}((I - F) + (L - \mathrm{FL}) + (T - \mathrm{FT}) + (\mathrm{LT} - \mathrm{FLT})), \\
D_y s_f(w) &= \frac{1}{(4h)}((I - L) + (F - \mathrm{FL}) + (T - \mathrm{LT}) + (\mathrm{FT} - \mathrm{FLT})), \qquad (7.1) \\
D_z s_f(w) &= \frac{1}{(4h)}((I - T) + (F - \mathrm{FT}) + (L - \mathrm{LT}) + (\mathrm{FL} - \mathrm{FLT})).
\end{aligned}
$$

TABLE 4 *Approximation of $D_x(f)$ by $D_x(s_f)$*

$1/h$	$\mathrm{err}_{\mathrm{mean}}^{D_x(f)}$	$\mathrm{err}_{\mathrm{rms}}^{D_x(f)}$	$\mathrm{err}_{\mathrm{max}}^{D_x(f)}$	$\mathrm{err}_{\mathrm{data}}^{D_x(f)}$
16	0.0217446	0.0357819	0.2247530	0.1916200
32	0.0054486	0.0090164	0.0603435	0.0496082
64	0.0013590	0.0022565	0.0152764	0.0125555
128	0.0003350	0.0005582	0.0038339	0.0031441
256	0.0000836	0.0001394	0.0009591	0.0007870

(ii) For each vertex v of V, we have

$$\begin{aligned}
D_x s_f(v) &= \frac{1}{(4h)}\left(\frac{3}{4}(B-F)+\frac{1}{16}((\mathrm{BL}-\mathrm{FL})\right.\\
&\qquad \left. +(\mathrm{BR}-\mathrm{FR})+(\mathrm{BT}-\mathrm{FT})-(\mathrm{BD}-\mathrm{FD}))\right),\\
D_y s_f(v) &= \frac{1}{(4h)}\left(\frac{3}{4}(R-L)+\frac{1}{16}((\mathrm{FR}-\mathrm{FL})\right.\\
&\qquad \left. +(\mathrm{BR}-\mathrm{BL})+(\mathrm{RT}-\mathrm{LT})-(\mathrm{RD}-\mathrm{LD}))\right),\\
D_z s_f(v) &= \frac{1}{(4h)}\left(\frac{3}{4}(T-D)+\frac{1}{16}((\mathrm{FT}-\mathrm{FD})\right.\\
&\qquad \left. +(\mathrm{BT}-\mathrm{BD})+(\mathrm{RT}-\mathrm{RD})+(\mathrm{LT}-\mathrm{LD}))\right).
\end{aligned} \tag{7.2}$$

Note that (7.1) and (7.2) are approximate derivatives naturally obtained from the data. More precisely, we observe that these derivatives are automatically determined as averages of weighted differences of the local gridded data. Besides the fact that we can use the derivatives of s_f directly, in contrast to standard approaches (see Marschner & Lobb, 1994; Meissner *et al.*, 2000; Parker *et al.*, 1998) of approximating derivatives (to set up the corresponding, independent trilinear models), no information from certain intermediate samples remains unused in our scheme.

Additionally, formulae (7.1) and (7.2) show that at each vertex of $\diamondsuit$ and V, the spline s_f satisfies relations of Hermite interpolation type. More precisely, s_f interpolates the approximative derivatives obtained from the given data values. However, no operator involving Hermite interpolation in the classical sense (prescribing the value and the three partial derivatives, independently) at these points exists because of some structural properties of $\mathcal{S}$ (see Hangelbroek *et al.*, 2004; Schumaker & Sorokina, 2005).

REMARK 7.2 When compared with well-known univariate spline approximation methods (see de Boor & Fix, 1973; Lyche & Schumaker, 1975; Marsden, 1970; Schoenberg, 1967, and references therein), our approach is related to the idea behind Schoenberg's operator (Schoenberg, 1967) and the more general approaches in Lyche & Schumaker (1975) involving point evaluation functionals. The connection can be seen easily when Schoenberg's spline approximant based on the cubic C^1 spline with double knots is written in its piecewise BB-form. As for a bivariate analogue of our scheme, we refer the reader to

Sorokina & Zeilfelder (2005). The advantage of these constructions is that they require no (approximate) derivatives at any point of the domain, which is also the case for our quasi-interpolation scheme. However, our approach differs from the univariate spline methods since we do not use any basis of the spline space and work with the polynomial pieces directly. The algorithm for developing the BB-coefficients formulae (3.3)–(3.9) comes from repeatedly averaging the data so that the approximation order is preserved. The averaging weights in (3.2) are obtained by satisfying the smoothness requirements in (2.6)–(2.9) simultaneously.

REMARK 7.3 If the data of f are given only at the points v_{ijk}, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, r$, then our scheme requires an extension of the data to the larger domain $\tilde{\Omega}$. This can be done in several ways. A natural example preserving the polynomials in span$\{1, x, y, z, xy, xz, yz, xyz\}$ is to define the missing data successively as follows:

$$\begin{aligned} &f_{0jk} := 2 f_{1jk} - f_{2jk}, \quad j = 1, \ldots, m, \ \ k = 1, \ldots, r, \\ &f_{00k} := 2 f_{01k} - f_{02k}, \quad f_{0m+1k} := 2 f_{0mk} - f_{0m-1k}, \quad k = 1, \ldots, r, \\ &f_{0j0} := 2 f_{0j1} - f_{0j2}, \quad f_{0jr+1} := 2 f_{0jr} - f_{0jr-1}, \quad j = 0, \ldots, m+1, \end{aligned} \tag{7.3}$$

with analogous settings for the remaining missing data values.

REMARK 7.4 Due to the uniform structure of the underlying tetrahedral partition and the symmetry of our quasi-interpolation scheme, it is possible to compute constants in Theorems 6.1–6.3 explicitly. This can be done with the help of a Markov type inequality with explicit constants. More precisely, using (2.4) and the techniques of Farin (1986) to compute the derivatives of polynomials in its BB-form, it can be shown that for any $p \in \mathcal{P}_3$ on $T \in \triangle$

$$\|D_x^{\alpha} D_y^{\alpha} D_z^{\alpha} p\|_T \leqslant M_{\alpha\beta\gamma} \|p\|_T, \quad \text{where } M_{\alpha\beta\gamma} = \begin{cases} 94, & \text{if } \alpha+\beta+\gamma = 1, \\ 1152, & \text{if } \alpha+\beta+\gamma = 2. \end{cases}$$

The arguments from the proofs of Theorems 6.1 and 6.2 show, for instance, that if $f \in C^2(\tilde{\Omega})$ then

$$\|f - \mathcal{Q}(f)\|_{\Omega} \leqslant 21 \|D^2 f\|_{\tilde{\Omega}} h^2,$$

and if $f \in C^3(\tilde{\Omega})$ then

$$\begin{aligned} &\|D_x^{\alpha} D_y^{\beta} D_z^{\gamma} (f - \mathcal{Q}(f))\|_{\Omega} \leqslant 1.5 \times 10^3 \|D^3 f\|_{\tilde{\Omega}} h^2, \quad \alpha+\beta+\gamma = 1, \\ &\|D_x^{\alpha} D_y^{\beta} D_z^{\gamma} (f - \mathcal{Q}(f))\|_{\Omega} \leqslant 17.5 \times 10^3 \|D^3 f\|_{\tilde{\Omega}} h, \quad \alpha+\beta+\gamma = 2. \end{aligned}$$

Moreover, we note that the constants slightly increase if we use the data described in Remark 7.3.

REMARK 7.5 We have also applied our scheme to simple test functions

$$f_1(x, y, z) = 1 + x + y + z + xy + xz + yz + xyz \quad \text{and} \quad f_2(x, y, z) = x^2,$$

and verified the results of Lemmas 5.1 and 5.2 numerically. More precisely, the numerical tests for these functions show that the values of $f_1 - s_{f_1}$ as well as the corresponding derivatives are numerical zeros, while the error $f_2 - s_{f_2}$ is at most $h^2/4$, and the corresponding derivatives of this function are also numerical zeros.

REMARK 7.6 Our scheme can be extended to more general partitions, where the grid points are spaced non-uniformly, i.e.

$$V := \{v_{ijk} = (ih_i, jh_j, kh_k), i = 0, \ldots, n+1, j = 0, \ldots, m+1, k = 0, \ldots, r+1\}.$$

In this case, the weights $\omega_{i_0 j_0 k_0}$ in (3.2) have to be adjusted and the norm of the corresponding quasi-interpolation operator changes.

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